Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 5, pp. 734–736, 2007

FUNCTIONAL RELATION
BETWEEN TWO SYMMETRIC SECOND-RANK TENSORS

N. I. Ostrosablin

UDC 539.3 : 517.958

The functional relationship between two symmetric second-rank tensors is considered. A new interpretation of the components of the tensors as projections onto an orthogonal tensor basis is given. It is shown that the constitutive relations can be written in the form of six functions each of which depends on one variable.

Key words: *symmetric tensors, tensor basis, orthogonal matrices, constitutive relations.*

The functional relationship between two symmetric second-rank tensors, which is important in constructing constitutive relations in continuum mechanics, has been the subject of many studies (see, e.g., [1–7]). It is known that in the space of symmetric second-rank tensors there exists an orthonormal basis of six tensors [3, 8–10]

$$t_{ijpq} = t_{jipq}, \qquad t_{ijpq} = t_{ijqp}, \qquad t_{ijpq}t_{ijrs} = \delta_{pqrs} = (\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr})/2,$$

where $\delta_{pr} = 1$ for $p = r$ and $\delta_{pr} = 0$ for $p \neq r$. In the present paper, Cartesian rectangular coordinates x_1, x_2, and x_3 are used and the summation is performed over repeated indices. The first two indices denote the tensor components and the last two denote the tensor number, the tensors with numbers pq and qp being identical. We use the following contracted notation for the indices of the symmetric tensors $h_{ij} = h_{ji}$:

$$\begin{gathered} h_1 = h_{11}, \qquad h_2 = h_{22}, \qquad h_3 = h_{33}, \\ h_4 = \sqrt{2}\, h_{23}, \qquad h_5 = \sqrt{2}\, h_{13}, \qquad h_6 = \sqrt{2}\, h_{12}. \end{gathered} \tag{1}$$

Then, the tensors t_{ijpq} correspond to an arbitrary orthogonal matrix of the sixth order t_{ip} $(i, p = \overline{1, 6})$; $t_{ip}t_{iq} = \delta_{pq}$. The orthogonal matrix t_{ip} depends on 15 free parameters c_{ip} $(i > p)$ and is obtained by orthonormalization of an arbitrary triangular matrix [9]

$$[c_{ip}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ c_{21} & 1 & 0 & 0 & 0 & 0 \\ c_{31} & c_{32} & 1 & 0 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & 1 & 0 & 0 \\ c_{51} & c_{52} & c_{53} & c_{54} & 1 & 0 \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & 1 \end{bmatrix}.$$

The stress tensor σ_{ij} and the strain tensor ε_{ij} are decomposed using the orthogonal basis t_{ip} [8–10]:

$$\sigma_i = t_{ip}\tilde{\sigma}_p, \qquad \tilde{\sigma}_p = \sigma_i t_{ip}, \qquad \varepsilon_i = t_{iq}\tilde{\varepsilon}_q, \qquad \tilde{\varepsilon}_q = \varepsilon_i t_{iq}. \tag{2}$$

Here and below, the contracted notation (1) is used. According to (2), the quantities $\tilde{\sigma}_p$ and $\tilde{\varepsilon}_q$ are projections of the tensors σ_i and ε_i onto the tensors of the orthogonal basis t_{ip} $(p = \overline{1, 6})$, i.e., convolutions of two symmetric second-rank tensors. It follows that the quantities $\tilde{\sigma}_p$ and $\tilde{\varepsilon}_q$ remain invariant upon orthogonal transformation of the coordinate system x_i.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; abd@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 5, pp. 134–137, September–October, 2007. Original article submitted October 26, 2006.

 0021-8944/07/4805-0734

Il'yushin [1, 2] considered rotations and reflections of the form

$$\tilde{\varepsilon}_p = \alpha_{pq}\tilde{\varepsilon}^*_q, \qquad \tilde{\varepsilon}^*_q = \tilde{\varepsilon}_p\alpha_{pq} \qquad (p, q = \overline{1, 6}), \tag{3}$$

where α_{pq} is an arbitrary orthogonal matrix of the sixth order ($\alpha_{pq}\alpha_{pr} = \delta_{qr}$). The matrix α_{pq} is determined by 15 independent parameters [9]. Since

$$\tilde{\varepsilon}_p = \varepsilon_i t_{ip}, \qquad \tilde{\varepsilon}^*_q = \varepsilon_i t^*_{iq}, \tag{4}$$

where t^*_{iq} is an orthogonal basis in the space of symmetric second-rank tensors different from the basis t_{ip}, from (3) and (4) we obtain

$$\varepsilon_i t_{ip} = \alpha_{pq}\varepsilon_i t^*_{iq}, \qquad \varepsilon_i(t_{ip} - \alpha_{pq}t^*_{iq}) = 0;$$

$$t_{ip} = t^*_{iq}\alpha_{pq}, \qquad t^*_{iq} = t_{ip}\alpha_{pq}. \tag{5}$$

Thus, transformation (3) corresponds to the orthogonal transformation of the basis (5).

Since the choice of the basis t_{ip} or t^*_{iq} is arbitrary, the functional relationship between $\tilde{\sigma}_p$ and $\tilde{\varepsilon}_q$ should not depend on transformations of the form of (3) (compare with [1, 2]):

$$\tilde{\sigma}_p = f_p(\tilde{\varepsilon}_q), \qquad \tilde{\sigma}^*_p = f^*_p(\tilde{\varepsilon}^*_q), \qquad p, q = \overline{1, 6}. \tag{6}$$

It should be noted that there is, in essence, no difference between σ_i and $\tilde{\sigma}_p$ (or between ε_i and $\tilde{\varepsilon}_p$). Indeed, t_{ip} ($p = \overline{1, 6}$) are six second-rank tensors which form an orthogonal basis in the space of symmetric tensors. The matrix t_{ip} can be an arbitrary orthogonal matrix, in particular, a unit matrix ($t_{ip} = \delta_{ip}$). Then,

$$\tilde{\sigma}_p = \sigma_i t_{ip} = \sigma_i\delta_{ip} = \sigma_p, \qquad \sigma_i = \delta_{ip}\tilde{\sigma}_p = \tilde{\sigma}_i. \tag{7}$$

Thus, from (7) it follows that the quantities σ_i and $\tilde{\sigma}_p$ differ only in notation. However, the representation $\tilde{\sigma}_p = \sigma_i\delta_{ip}$ implies that $\tilde{\sigma}_p = \sigma_p$ are invariants and projections onto the basis $\delta_{i1}, \delta_{i2}, \ldots, \delta_{i6}$. This basis can be written as follows:

$$\delta_{ij11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \delta_{ij22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \delta_{ij33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\sqrt{2}\,\delta_{ij23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \sqrt{2}\,\delta_{ij13} = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad \sqrt{2}\,\delta_{ij12} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This interpretation of the tensors, different from the traditional interpretation, was first proposed in [11, 12]. It follows that the constitutive relations generally have the form of (6):

$$\tilde{\sigma}_p = f_p(\tilde{\varepsilon}_q) \qquad (p, q = \overline{1, 6}), \tag{8}$$

i.e., they functionally link the invariants $\tilde{\sigma}_p$ and $\tilde{\varepsilon}_q$.

As the coordinate system x_i, the basis t_{ip} is chosen from considerations of simplicity and convenience. For elastic anisotropic materials, t_{ip} are eigenstates. In this case, Hooke's law [8–10] becomes

$$\tilde{\sigma}_p = \lambda_{pq}\tilde{\varepsilon}_q, \qquad p = q. \tag{9}$$

The basis t_{ip} should be chosen in such a manner that the constitutive relations (8) have the simplest form [for example, they can be written in the form of (9)]:

$$\tilde{\sigma}_1 = f_1(\tilde{\varepsilon}_1), \qquad \tilde{\sigma}_2 = f_2(\tilde{\varepsilon}_2), \qquad \ldots, \qquad \tilde{\sigma}_6 = f_6(\tilde{\varepsilon}_6), \tag{10}$$

i.e., each function depends on one variable.

We write relation (8) as $\sigma_i t_{ip} = f_p(\varepsilon_i t_{iq})$. Let $t_{ip} = t^*_{iq}\alpha_{pq}$, then

$$\sigma_i t^*_{ip}\alpha_{pq} = f_p(\varepsilon_i t^*_{ir}\alpha_{sr}), \qquad \tilde{\sigma}^*_q\alpha_{pq} = f_p(\tilde{\varepsilon}^*_r\alpha_{sr});$$

$$\tilde{\sigma}^*_q = f_p(\tilde{\varepsilon}^*_r\alpha_{sr})\alpha_{pq} = f^*_q(\tilde{\varepsilon}^*_r) \tag{11}$$

or, taking into account (3), we obtain

$$\tilde{\sigma}_p = f_p(\tilde{\varepsilon}_s), \qquad \alpha_{pq}\tilde{\sigma}^*_q = f_p(\alpha_{sr}\tilde{\varepsilon}^*_r);$$

$$\tilde{\sigma}^*_q = \alpha_{pq} f_p(\alpha_{sr}\tilde{\varepsilon}^*_r) = f^*_q(\tilde{\varepsilon}^*_r), \tag{12}$$

i.e., (12) is identical to (11).

Thus, when the basis (5) is changed, the constitutive relations (8) are transformed according to formulas (11) and (12).

From (10) and (3), we obtain

$$\alpha_{1q}\tilde{\sigma}^*_q = f_1(\alpha_{1r}\tilde{\varepsilon}^*_r), \qquad \alpha_{2q}\tilde{\sigma}^*_q = f_2(\alpha_{2r}\tilde{\varepsilon}^*_r), \qquad \alpha_{3q}\tilde{\sigma}^*_q = f_3(\alpha_{3r}\tilde{\varepsilon}^*_r),$$

$$\alpha_{4q}\tilde{\sigma}^*_q = f_4(\alpha_{4r}\tilde{\varepsilon}^*_r), \qquad \alpha_{5q}\tilde{\sigma}^*_q = f_5(\alpha_{5r}\tilde{\varepsilon}^*_r), \qquad \alpha_{6q}\tilde{\sigma}^*_q = f_6(\alpha_{6r}\tilde{\varepsilon}^*_r),$$

whence we have

$$\tilde{\sigma}^*_q = \alpha_{1q} f_1(\alpha_{1r}\tilde{\varepsilon}^*_r) + \alpha_{2q} f_2(\alpha_{2r}\tilde{\varepsilon}^*_r) + \alpha_{3q} f_3(\alpha_{3r}\tilde{\varepsilon}^*_r) + \alpha_{4q} f_4(\alpha_{4r}\tilde{\varepsilon}^*_r) + \alpha_{5q} f_5(\alpha_{5r}\tilde{\varepsilon}^*_r) + \alpha_{6q} f_6(\alpha_{6r}\tilde{\varepsilon}^*_r), \tag{13}$$

i.e., changing the basis from t_{ip} to $t^*_{ip} = t_{ip}\alpha_{pq}$, one can pass from (10) to (13) and vice versa.

In summary, if the constitutive relations are written in the form of (13), they can be reduced to the form of (10), in which each function depends on one variable. An arbitrary basis can be obtained from a specific basis using formulas (5).

This work was supported by the Russian Foundation for Basic Research (Grant No. 05-01-00728).

REFERENCES

1. A. A. Il'yushin, "On the relationship between stresses and small strains in continuum mechanics," *Prikl. Mat. Mekh.*, **18**, No. 6, 641–666 (1954).
2. A. A. Il'yushin, *Plasticity. Principles of the General Mathematical Theory* [in Russian], Izd. Akad. Nauk SSSR, Moscow (1963).
3. V. V. Novozhilov, "The relationship between stresses and strains in initially isotropic inelastic solids (geometrical aspect)," *Prikl. Mat. Mekh.*, **27**, No. 5, 794–812 (1963).
4. K. F. Chernykh, "On the forms of relationship between symmetric tensors in the mechanics of continuous media," *Inzh. Zh., Mekh. Tverd. Tela.*, No. 3, 42–51 (1967).
5. K. F. Chernykh, "On the functional relationships between coaxial symmetric second-rank tensors," in: *Problems of the Mechanics of Deformable Solids* [in Russian], Sudostroenie, Leningrad (1970), pp. 481–486.
6. B. D. Annin, "Anisotropic tensor functions," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 11, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1972), pp. 94–97.
7. D. V. Georgievskii, "Tensor-nonlinear effects in the isothermal deformation of continuous media," *Usp. Mekh.*, **1**, No. 2, 150–176 (2002).
8. Ya. Rykhlevskii, "On Hooke's law," *Prikl. Mat. Mekh.*, **48**, No. 3, 420–435 (1984).
9. N. I. Ostrosablin, "Structure of the elastic tensor. Elastic eigenstates," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 66, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1984), pp. 113–125.
10. N. I. Ostrosablin, "On the structure of the elastic tensor and the classification of anisotropic materials," *j. Appl. Mech. Tech. Phys.*, **27**, No. 4, 600–607 (1986).
11. N. I. Ostrosablin, "Invariants of the fourth-rank elastic tensor," *Sib. Zh. Indust. Mat.*, **1**, No. 1, 155–163 (1998).
12. N. I. Ostrosablin, "Functional relationship between two symmetric second-rank tensors," in: *Deformation and Fracture of Structurally Inhomogeneous Media and Structures*, Abstracts of All-Russian Conf. (Novosibirsk, Russia, October 9–13, 2006), Novosibirsk (2006), p. 96.